Electrical and Electronics Engineering
2024-2025
Master Semester 2

Course
Smart grids technologies
Approximations of the Optimal Power Flow problem

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Outline

Introduction

The DC approximation

Linearized OPF

Sequential linear programming

Introduction

The load flow equations written by means of the **grid nodal model** defines a **non-convex set of constraints for the OPF**.

$[\bar{\mathbf{I}}] = [\bar{\mathbf{Y}}][\bar{\mathbf{V}}]$ Kirchoff's laws (KVL and KCL) applied to the whole grid;

$\bar{S}_i = \bar{V}_i \underline{I}_i$ nodal power injection $\forall i = 1, \dots, s$;

$\sum_{i=1}^{s} \bar{S}_i = 0$ power balance over the whole grid.

Recall the load flow equations in **polar coordinates**

$$\begin{cases} P_i = \sum_{\ell=1}^{s} V_i V_\ell Y_{i\ell} \cos\left(\vartheta_i - \vartheta_\ell - \gamma_{i\ell}\right) \\ Q_i = \sum_{\ell=1}^{s} V_i V_\ell Y_{i\ell} \sin\left(\vartheta_i - \vartheta_\ell - \gamma_{i\ell}\right) \end{cases}$$

or in **rectangular coordinates**

$$\begin{cases} P_i = V_i' \sum_{\ell=1}^{s} \left(G_{i\ell} V_\ell' - B_{i\ell} V_\ell''\right) + V_i'' \sum_{\ell=1}^{s} \left(B_{i\ell} V_\ell' + G_{i\ell} V_\ell''\right) \\ Q_i = -V_i' \sum_{\ell=1}^{s} \left(B_{i\ell} V_\ell' + G_{i\ell} V_\ell''\right) + V_i'' \sum_{\ell=1}^{s} \left(G_{i\ell} V_\ell' - B_{i\ell} V_\ell''\right) \end{cases}$$

$$V_i^2 = V_i'^2 + V_i''^2$$

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The DC approximation

The non-convexity of the OPF constraints can be addressed by replacing them with a set of **approximated ones**.

The simplest one is the **DC-approximation** we have already seen in the module on the load flow.

IMPORTANT: any approximation of the OPF constraints implies that the solution of the problem does not satisfy the load flow equations. So, the violation of the constraints (may) still exist if we compute the load flow from the obtained solution of the approximated OPF.

The DC approximation (ref. lecture 2.4)

- **For high voltage systems, the longitudinal resistances of the line conductors and copper losses of transformers are neglected with respect to the series reactance of the lines and transformers (note that this is not true in distribution systems).**
  - Acceptable when the calculation of losses is waived.
  - $\frac{x}{r} \approx 10$ for transmission lines,
  - $\frac{x}{r} \approx 50$ for transformers
  - In both cases $\bar{z} \approx jx$
- **The transverse admittances of the network components are neglected.**
  - The shunt capacitances of the lines generate reactive power especially in long lines at very high voltage;
  - The currents flowing through the shunt capacitances are mainly associated to the reactive power balance of the line and, in high voltage systems, they are related to the difference of voltages magnitudes at the extremes of the lines.
  - In high voltage systems, however, shunt capacitances have a little influence on the active power flows that mainly depend on the differences between the phases of the voltage phasors at the line ends.
  - The shunt conductance, which take into account the corona and insulators losses of the lines and the iron-losses transformers, may assumed small and, in the DC approximation, negligible.

With these simplifications, **the grid model is only composed by longitudinal inductive reactances** (i.e., the equivalent series reactance of lines and transformers).

The DC approximation (ref. lecture 2.4)

- $\bar{V}_i$, $\bar{V}_l$ are the voltages phasors at the extremities
- θ_i is the argument of $\bar{V}_i$ and θ_l the argument of $\bar{V}_l$, $\theta_{il} = \theta_i - \theta_l$
- x_{il} the reactance of the branch il

In view of the above, the active power through the branch il is:

$$P_{il} = \frac{3V_iV_l}{x_{il}} \sin\theta_{il}$$

Further hypotheses of the DC approximation are the following:

- the modules of the nodal voltages all equal to $1\ pu$
- the difference $\theta_i - \theta_l$ is small, therefore $sin(\theta_i - \theta_l) \approx \theta_i - \theta_l$

Therefore, we have

$$P_{il} = \frac{1}{x_{il}} \theta_{il}$$

As a consequence, the injection of power in a generical node i is :

$$P_i = \sum_{l \neq i} P_{il} = \frac{\theta_{i1}}{x_{i1}} + \cdots \frac{\theta_{is}}{x_{is}}$$

(note that the voltage angles are known up to one phase shift, so we need to take one node, say k, as reference and set $\theta_k = 0$ at this node), so we have that:

$$P_i = \left(\frac{1}{x_{i1}} + \cdots \frac{1}{x_{is}}\right) \theta_i - \sum_{\substack{l=1 \\ l \neq i}}^{s} \frac{1}{x_{il}} \theta_l = \sum_{l=1}^{s} B_{il}\theta_l$$

The DC approximation (ref. lecture 2.4)

The following linear matrix equation is obtained for the whole network:

$$[P] = [B] \times [\theta]$$

where $[B]$ and the "susceptance matrix" of the entire transmission network (in pu).

As seen before, the diagonal terms B_{ii} of $[B]$ consist of the sum of the (longitudinal) susceptance of all sides converging at the i-th node

The terms on the diagonal B_{ii} are positive if the susceptance are inductive while the other terms are all negative, provided that the susceptance are inductive, and meet condition

$$B_{lk} = B_{kl}$$

The DC approximation

Let's apply the DC load flow approximation to our 3-bus example.

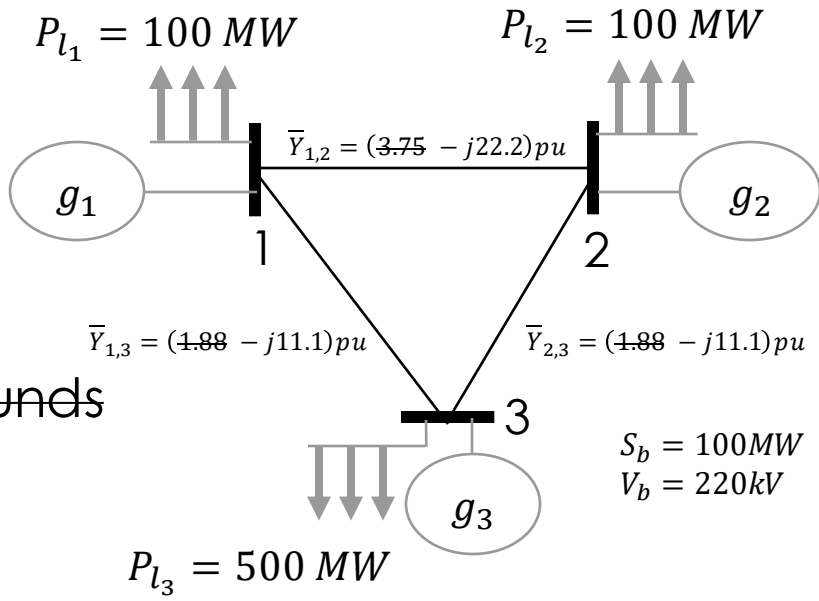
Cost function: $\sum_{i=1,3} C_i\left(P_{g_i}\right)$

Decision variables:

- $P_{g_2}, P_{g_3}, \sout{Q_{g_2}, Q_{g_3}}$

Constraints:

- DC grid's load flow equations
- ~~Nodal voltage magnitudes within bounds~~
- Branches powers below max
- Generators P^{min}, P^{max} ~~$and\ Q^{min}, Q^{max}$~~.



Quantity	Value
$P_{g_i}^{min}, P_{g_i}^{max}$	$0 \div 400\ MW$
~~$Q_{g_i}^{min}, Q_{g_i}^{max}$~~	~~$-80 \div +80\ MVar$~~
C_1, C_2, C_3	$15, 1, 225\ CHF/MWh$
$S_{12}^{max}, S_{23}^{max}, S_{31}^{max}$	$200, 200, 300\ MW$ ~~MVA~~

The DC approximation

Let's apply the DC load flow approximation to our 3-bus example.

$$P_1 = 22.2\,(\theta_1 - \theta_2) + 11.1(\theta_1 - \theta_3)$$

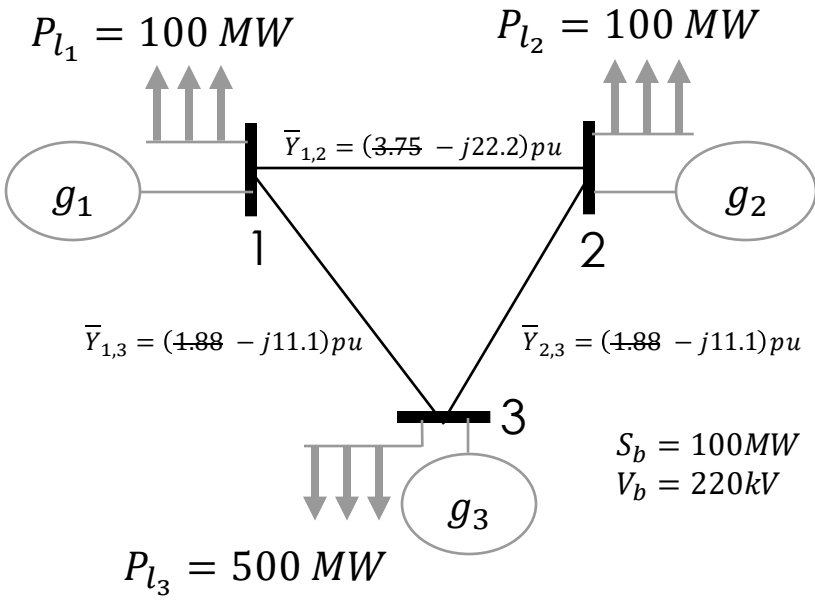
$$P_2 = 22.2\,(\theta_2 - \theta_1) + 11.1(\theta_2 - \theta_3)$$

$$P_3 = 11.1(\theta_3 - \theta_1) + 11.1\,(\theta_3 - \theta_2)$$

$$P_1 = P_{g_1} + P_{l_1}$$

$$P_2 = P_{g_2} + P_{l_2}$$

$$P_3 = P_{g_3} + P_{l_3}$$



If nodal injections (P_1, P_2, P_3) are given such that $P_1 + P_2 + P_3 = 0$, we can solve for θ.

Let us set $\theta_1 = 0$ (reference bus = slack bus).

The branch flows are derived from θ as:

$$P_{1,2} = 22.2\,(-\theta_2)$$

$$P_{1,3} = 11.1(-\theta_3)$$

$$P_{2,3} = 11.1(\theta_2 - \theta_3)$$

The DC approximation

The OPF with the DC approximation has **linear constraints**, so the **feasible set of the problem is convex**, and a **convex cost function**. So, it is a **convex optimisation problem**.

$$\min_{P_{g_2},P_{g_3}} \sum_{i=1}^{3} C_i\left(P_{g_i}\right)$$

$$s.t.$$

$$P_1 = 22.2\,(-\theta_2) + 11.1(-\theta_3)$$

$$P_2 = 22.2\,(\theta_2) + 11.1(\theta_2 - \theta_3)$$

$$P_3 = 11.1(\theta_3) + 11.1\,(\theta_3 - \theta_2)$$

$$P_1 = P_{g_1} + P_{l_1}$$

$$P_2 = P_{g_2} + P_{l_2}$$

$$P_3 = P_{g_3} + P_{l_3}$$

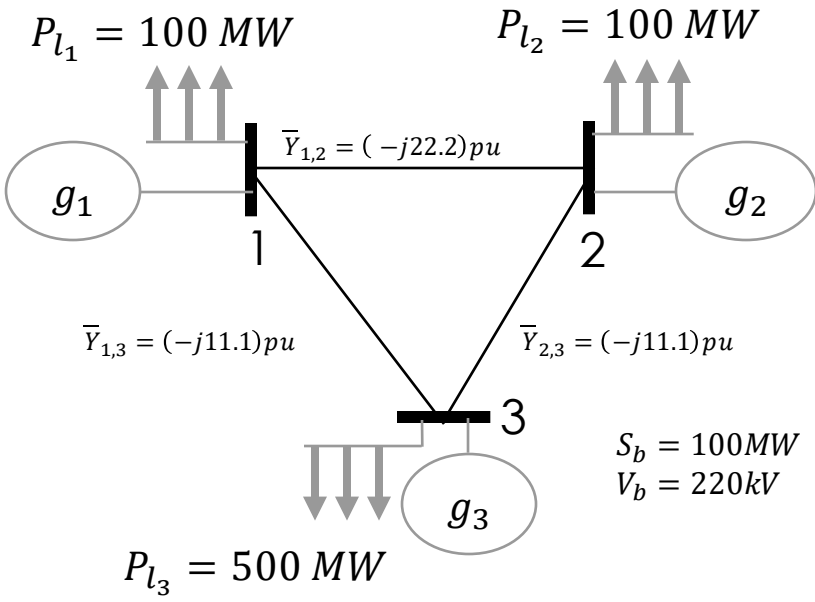
$$P_{g_i}^{min} \le P_{g_i} \le P_{g_i}^{max},\ i = 1,2,3$$

$$-P_{1,2}^{max} \le 22.2\,(0 - \theta_2) \le P_{1,2}^{max}$$

$$-P_{1,3}^{max} \le 11.1(0 - \theta_3) \le P_{1,3}^{max}$$

$$-P_{2,3}^{max} \le 11.1(\theta_2 - \theta_3) \le P_{2,3}^{max}$$

$$-\pi \le \theta_i \le \pi,\ i = 2,3$$



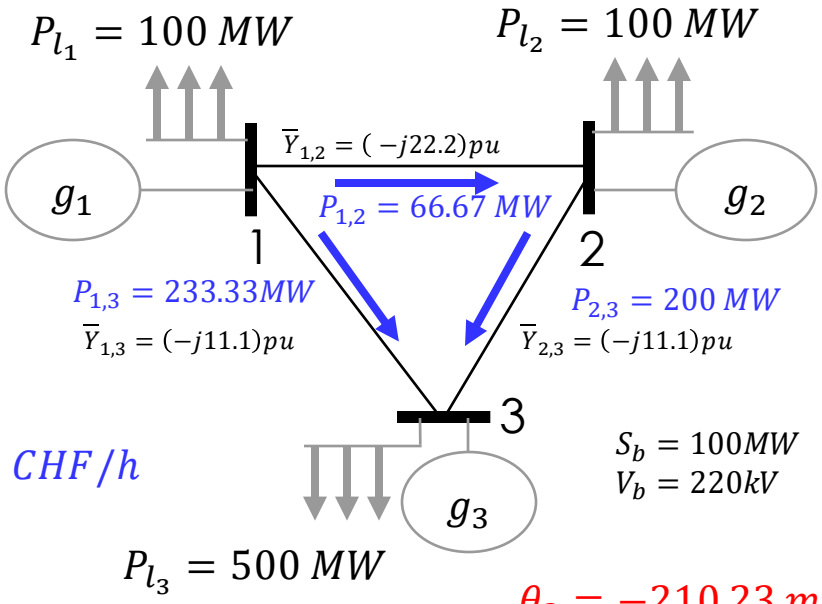
Quantity	Value
$P_{g_i}^{min}, P_{g_i}^{max}$	$0 \div 400\ MW$
C_1, C_2, C_3	$15, 1, 225\ CHF/MWh$
$S_{12}^{max}, S_{23}^{max}, S_{31}^{max}$	$200, 200, 300\ MW$

The DC approximation

Solution

$\theta_1 = 0\ mrad$
$\lambda_{P_1} = 75.67\ CHF/MWh$
$P_{g_1} = 400 MW$

$\theta_2 = -30.03\ mrad$
$\lambda_{P_2} = 1\ CHF/MWh$
$P_{g_2} = 233.33\ MW$



$S_b = 100MW$
$V_b = 220kV$

Total cost: 21233 CHF/h

$\theta_3 = -210.23\ mrad$
$\lambda_{P_3} = 225\ CHF/MWh$
$P_{g_3} = 66.67\ MW$

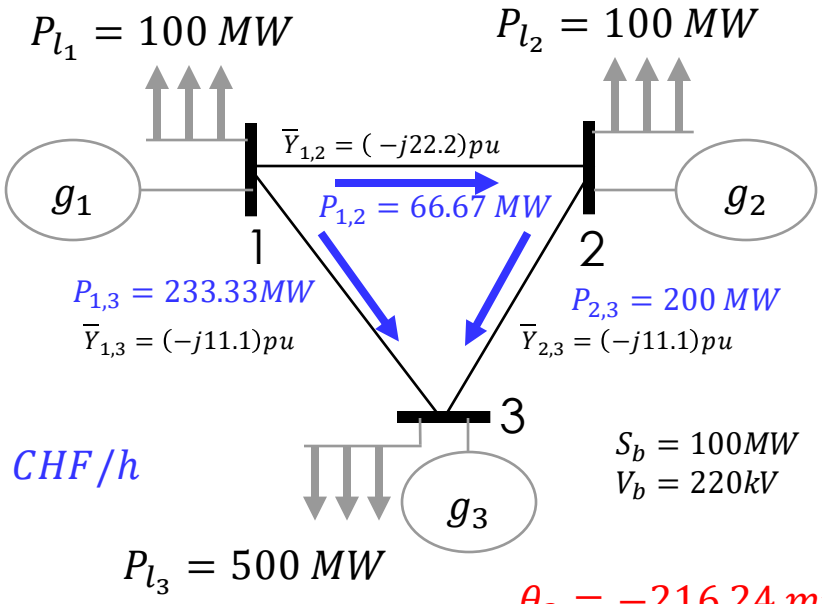
Quantity	Value
$P_{g_i}^{min}, P_{g_i}^{max}$	$0 \div 400\ MW$
C_1, C_2, C_3	$15, 1, 225\ CHF/MWh$
$S_{12}^{max}, S_{23}^{max}, S_{31}^{max}$	$200, 200, 300\ MW$

The DC approximation

Solution

$\theta_1 = 0\ mrad$
$\lambda_{P_1} = 15\ CHF/MWh$
$P_{g_1} = 300 MW$

$\theta_2 = +18.03\ mrad$
$\lambda_{P_2} = 15\ CHF/MWh$
$P_{g_2} = 400\ MW$



$Total\ cost:\ 13767\ CHF/h$

$\theta_3 = -216.24\ mrad$
$\lambda_{P_3} = 15\ CHF/MWh$
$P_{g_3} = 0\ MW$

Quantity	Value
$P_{g_i}^{min}, P_{g_i}^{max}$	$0 \div 400\ MW$
C_1, C_2, C_3	$15, 1, 225\ CHF/MWh$
$S_{12}^{max}, S_{23}^{max}, S_{31}^{max}$	$2000, 2000, 3000\ MW$

Lines power transmission limits x10

The DC approximation

From this numerical example, we can conclude that **the DC OPF problem produces results that are qualitatively like the solution of the original non-convex AC-OPF (see lecture 4.1)**, with the main difference that **active power losses are neglected as well as voltage magnitudes variations and reactive powers.**

Remember that these considerations are valid for **high voltage power systems** where the **DC approximation may hold**. In **power distribution systems (where the longitudinal impedance of branches is more resistive than inductive)**, or in the case where **reactive power flows have to be considered along with voltage constraints**, **the DC-OPF cannot be used**.

Outline

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The DC approximation

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Sequential linear programming OPF

Linearized OPF

Let us consider a grid with s buses where the slack is located at bus 1.

As always, we can express the generic power at bus i as

$$\overline{S}_i = \overline{V}_i \sum_{j=1}^{s} \underline{V}_j \underline{Y}_{ij}, i = 1, \dots, s$$

Given that we have fixed the voltage at the slack

$$\left|\overline{V}_1\right| = 1pu,\ \arg\left(\overline{V}_1\right) = 0$$

If we indicate with $\overline{\mathbf{S}} = \left(\overline{S}_2, \dots, \overline{S}_s\right)^T, \overline{\mathbf{V}} = \left(\overline{V}_2, \dots, \overline{V}_s\right)^T$ the **arrays of power injections (i.e., only PQ nodes compose the LF boundary conditions) and voltages in all the nodes except the slack**, we can write the load flow problem in a compact form as follows:

$$\overline{\mathbf{S}} = f\left(\overline{\mathbf{V}}\right), \qquad \overline{S}_1 = f_1\left(\overline{\mathbf{V}}\right)$$

where we compute $\left(\overline{\mathbf{V}}, \overline{S}_1\right)$ when $\overline{\mathbf{S}}$ is given (e.g. when we fix the nodal power injections and slack voltage) from the numerical inversion of f.

Linearized OPF

Let us assume that we know the state of the system (for instance from a state estimator). This means that we know $\overline{\mathbf{V}}^*$ for a given $\overline{\mathbf{S}}^*$.
Therefore, we can express $\overline{\mathbf{S}}$ in Taylor series truncated at the first order:

$$\overline{\mathbf{S}} = \overline{\mathbf{S}}^* + \nabla f\left(\overline{\mathbf{V}}^*\right)\left(\overline{\mathbf{V}} - \overline{\mathbf{V}}^*\right)$$

where $\nabla f\left(\overline{\mathbf{V}}^*\right)$ is the Jacobian of f computed in $\overline{\mathbf{V}}^*$.

The above equation is (trivially) linear and can be used to approximate the load flow non-convex constraints and make them convex.

IMPORTANT: remember that this approximation is accurate if $\overline{\mathbf{S}}$ and $\overline{\mathbf{V}}$ are close to $\overline{\mathbf{S}}^*$ and $\overline{\mathbf{V}}^*$.

Linearized OPF

It is interesting to note that $\nabla f\left(\overline{\mathbf{V}}^*\right)$ is composed by the partial derivatives of the nodal power injections with respect to the nodal voltages. **In other words, $\nabla f\left(\overline{\mathbf{V}}^*\right)$ is the Jacobian of the Newton-Raphson solution of the load flow problem.**

Let's write it for the case of polar coordinates.

$$\begin{bmatrix} P_2 \\ \cdots \\ P_s \\ Q_2 \\ \cdots \\ Q_s \end{bmatrix} \approx \begin{bmatrix} P_2^* \\ \cdots \\ P_s^* \\ Q_2^* \\ \cdots \\ Q_s^* \end{bmatrix} + \underbrace{\left[\begin{matrix} \frac{\partial P_2}{\partial V_2} & \cdots & \frac{\partial P_2}{\partial \theta_s} \\ \vdots & \ddots & \vdots \\ \frac{\partial Q_s}{\partial V_2} & \cdots & \frac{\partial Q_s}{\partial \theta_s} \end{matrix}\right]_{(\mathbf{V}^*,\boldsymbol{\theta}^*)}}_{\nabla f\left(\overline{\mathbf{V}}^*\right) = \mathbf{J}(\mathbf{V}^*,\boldsymbol{\theta}^*)} \begin{bmatrix} V_2 - V_2^* \\ \vdots \\ V_s - V_s^* \\ \theta_2 - \theta_2^* \\ \vdots \\ \theta_s - \theta_s^* \end{bmatrix}$$

Linearized OPF

By assuming that **the load flow converges in the surrounding of the state $\overline{\mathbf{V}}^*$ for a given $\overline{\mathbf{S}}^*$**, we have that the the **Jacobian $\nabla f\left(\overline{\mathbf{V}}^*\right)$ can be inverted**. Therefore, we can also express the nodal voltage variations as a function of the variations of the nodal power injections:

$$\begin{bmatrix} V_2 - V_2^* \\ \vdots \\ V_s - V_s^* \\ \theta_2 - \theta_2^* \\ \vdots \\ \theta_s - \theta_s^* \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial P_2}{\partial V_2} & \cdots & \frac{\partial P_2}{\partial \theta_s} \\ \vdots & \ddots & \vdots \\ \frac{\partial Q_s}{\partial V_2} & \cdots & \frac{\partial Q_s}{\partial \theta_s} \end{bmatrix}^{-1}\Bigg|_{(\mathbf{V}^*,\boldsymbol{\theta}^*)}}_{\mathbf{J}^{-1}(\mathbf{V}^*,\boldsymbol{\theta}^*)} \begin{bmatrix} P_2 - P_2^* \\ \vdots \\ P_s - P_s^* \\ Q_2 - Q_2^* \\ \vdots \\ Q_s - Q_s^* \end{bmatrix}$$

Where the components of the inverted Jacobian $\mathbf{J}^{-1}(\mathbf{V}^*,\boldsymbol{\theta}^*)$ are those that we have already computed in the lecture 2.4.

Linearized OPF

In the OPF **there is the need to compute the active and reactive powers at the slack** (i.e., $\overline{S}_1 = f_1(\overline{\mathbf{V}})$). This computation is rather simple since

$$\overline{S}_1 = P_1 + jQ_1 = \overline{V}_1 \sum_{j=1}^{s} \underline{V}_j \underline{Y}_{ij} = \sum_{j=1}^{s} V_1 V_j Y_{ij} e^{j(\theta_1 - \theta_j - \gamma_{il})}$$

$$P_1 = \sum_{j=1}^{s} V_1 V_j Y_{ij} cos(\theta_1 - \theta_j - \gamma_{il})$$

$$Q_1 = \sum_{j=1}^{s} V_1 V_j Y_{ij} sin(\theta_1 - \theta_j - \gamma_{il})$$

where $|\overline{V}_1|=1pu,\ \arg(\overline{V}_1) = \theta_1 = 0$, or $\overline{V}_1 = (1 + j0\ )pu$. Therefore, we got:

$$\begin{bmatrix} P_1 \\ Q_1 \end{bmatrix} \approx \begin{bmatrix} P_1^* \\ Q_1^* \end{bmatrix} + \left[\begin{matrix} \frac{\partial P_1}{\partial V_2} & \cdots & \frac{\partial P_1}{\partial \theta_s} \\ \frac{\partial Q_1}{\partial V_2} & \cdots & \frac{\partial Q_1}{\partial \theta_s} \end{matrix}\right]_{(\mathbf{V}^*, \boldsymbol{\theta}^*)} \begin{bmatrix} V_2 - V_2^* \\ \vdots \\ V_s - V_s^* \\ \theta_2 - \theta_2^* \\ \vdots \\ \theta_s - \theta_s^* \end{bmatrix}$$

Linearized OPF

Therefore, we can write the approximated linear OPF as follows

$$\min_{P_{g_2},\dots,P_{g_s},Q_{g_2},\dots,Q_{g_s}} \sum_{i=1}^{3} C_i\left(P_{g_i}, Q_{g_i}\right)$$

$$s.t.$$

$$\begin{bmatrix} P_2 \\ \dots \\ P_s \\ Q_2 \\ \dots \\ Q_s \end{bmatrix} \approx \begin{bmatrix} P_2^* \\ \dots \\ P_s^* \\ Q_2^* \\ \dots \\ Q_s^* \end{bmatrix} + \begin{bmatrix} \frac{\partial P_2}{\partial V_2} & \dots & \frac{\partial P_2}{\partial \theta_s} \\ \vdots & \ddots & \vdots \\ \frac{\partial Q_s}{\partial V_2} & \dots & \frac{\partial Q_s}{\partial \theta_s} \end{bmatrix}\Bigg|_{(\mathbf{V}^*,\boldsymbol{\theta}^*)} \begin{bmatrix} V_2 - V_2^* \\ \vdots \\ V_s - V_s^* \\ \theta_2 - \theta_2^* \\ \vdots \\ \theta_s - \theta_s^* \end{bmatrix}$$

$$\begin{bmatrix} P_1 \\ Q_1 \end{bmatrix} \approx \begin{bmatrix} P_1^* \\ Q_1^* \end{bmatrix} + \begin{bmatrix} \frac{\partial P_1}{\partial V_2} & \dots & \frac{\partial P_1}{\partial \theta_s} \\ \frac{\partial Q_1}{\partial V_2} & \dots & \frac{\partial Q_1}{\partial \theta_s} \end{bmatrix}\Bigg|_{(\mathbf{V}^*,\boldsymbol{\theta}^*)} \begin{bmatrix} V_2 - V_2^* \\ \vdots \\ V_s - V_s^* \\ \theta_2 - \theta_2^* \\ \vdots \\ \theta_s - \theta_s^* \end{bmatrix}$$

$$P_i = P_{g_i} + P_{l_i}, i = 1, \dots, s$$

$$Q_i = Q_{g_i} + Q_{l_i}, i = 1, \dots, s$$

$$P_{g_i}^{min} \le P_{g_i} \le P_{g_i}^{max},\ i = 1, \dots, g$$

$$Q_{g_i}^{min} \le Q_{g_i} \le Q_{g_i}^{max},\ i = 1, \dots, g$$

$$\left|\overline{V}_1\right| = 1pu,\ \arg\left(\overline{V}_1\right) = \theta_1 = 0;$$

$$V_{min} \le \left|\overline{V}_i\right| \le V_{max}, i = 2, \dots, s$$

$$-\pi \le \theta_i \le \pi, i = 2, \dots, s$$

This is a linear (so, convex) OPF. Note that in this approximated OPF the constraints on the branch flows are not included (see next slides how to fix this problem).

Linearized OPF

We can derive a simpler version of the problem by observing that the components of the inverted Jacobian $\mathbf{J}^{-1}(\mathbf{V}^*, \boldsymbol{\theta}^*)$ **are those that we have already computed in the lecture on the lecture 2.4** (we recall that $\mathcal{N}$ is the set of PQ buses, $\mathcal{H}$ the set of slack buses and that $\{1,2,\dots,s\} = \mathcal{H} \cup \mathcal{N},\ \mathcal{H} \cap \mathcal{N} = \emptyset$).

$$\mathbf{1}_{\{i=l\}} = \frac{\partial \underline{V}_i}{\partial P_l} \sum_{j \in \mathcal{H} \cup \mathcal{N}} \bar{Y}_{ij}\, \bar{V}^*{}_j + \underline{V}^*{}_i \sum_{j \in \mathcal{N}} \bar{Y}_{ij} \frac{\partial \bar{V}_j}{\partial P_l}$$

$$-j\mathbf{1}_{\{i=l\}} = \frac{\partial \underline{V}_i}{\partial Q_l} \sum_{j \in \mathcal{H} \cup \mathcal{N}} \bar{Y}_{ij}\, \bar{V}^*{}_j + \underline{V}^*{}_i \sum_{j \in \mathcal{N}} \bar{Y}_{ij} \frac{\partial \bar{V}_j}{\partial Q_l}$$

From the above **complex** sensitivity coefficients, we can compute the **nodal voltage module sensitivities:**

$$K^{il}_{P,V} = \frac{\partial |\bar{V}_i|}{\partial P_l} = \frac{1}{|\bar{V}_i|} \Re\left(\underline{V}_i \frac{\partial \bar{V}_i}{\partial P_l}\right)$$

$$K^{il}_{Q,V} = \frac{\partial |\bar{V}_i|}{\partial Q_l} = \frac{1}{|\bar{V}_i|} \Re\left(\underline{V}_i \frac{\partial \bar{V}_i}{\partial Q_l}\right)$$

Note that: $\bar{V}_i = V_i' + jV_i'',\ |\bar{V}_i| = \sqrt{(V_i')^2 + (V_i'')^2},\ \frac{\partial |\bar{V}_i|}{\partial P_l} = \frac{\partial \sqrt{(V_i')^2 + (V_i'')^2}}{\partial P_l} = \frac{1}{2\sqrt{(V_i')^2 + (V_i'')^2}} \frac{\partial \left((V_i')^2 + (V_i'')^2\right)}{\partial P_l} =$

$$\frac{1}{\sqrt{(V_i')^2 + (V_i'')^2}} \left(V_i' \frac{\partial V_i'}{\partial P_l} + V_i'' \frac{\partial V_i''}{\partial P_l}\right) = \frac{1}{|\bar{V}_i|} \Re\left(\underline{V}_i \frac{\partial \bar{V}_i}{\partial P_l}\right)$$

Linearized OPF

It is interesting to note that the operational constraints on the nodal voltage refers to their modules: $V_{min} \leq |\overline{V}_i| \leq V_{max}, i = 2, \ldots, s$.

Therefore, thanks to the sensitivity coefficients $K_P^{il} = \frac{\partial |\overline{V}_i|}{\partial P_l}$ and $K_Q^{il} = \frac{\partial |\overline{V}_i|}{\partial Q_l}$ we may write the constraints on the voltage modules as follows:

$$\begin{bmatrix} V_2 - V_2^* \\ \vdots \\ V_s - V_s^* \end{bmatrix} = [\mathbf{K}_{P,V}] \begin{bmatrix} P_2 - P_2^* \\ \vdots \\ P_s - P_s^* \end{bmatrix} + [\mathbf{K}_{Q,V}] \begin{bmatrix} Q_2 - Q_2^* \\ \vdots \\ Q_s - Q_s^* \end{bmatrix}$$

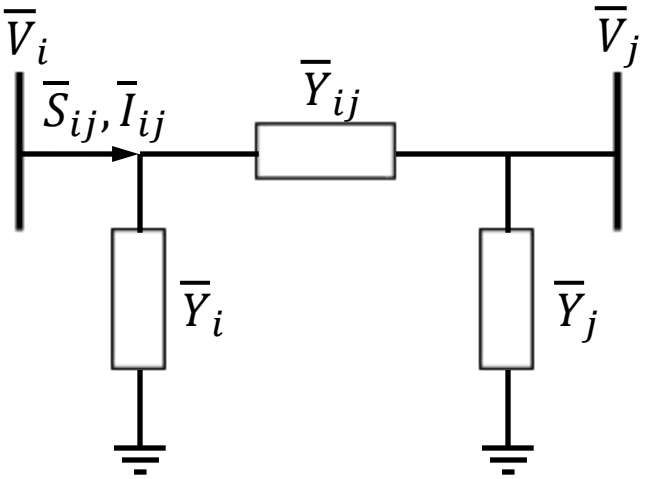
Note that in the above equation there is a slight abuse of nomenclature, where $V_i = |\overline{V}_i|$.

Linearized OPF

Furthermore, from the knowledge of the complex voltage sensitivity coefficients associated to the complex nodal voltages (i.e. $\frac{\partial \bar{V}_i}{\partial P_l}$ and $\frac{\partial \bar{V}_i}{\partial Q_l}$), we can easily **derive the sensitivity coefficients for the branch currents** as follows.

Let's express the branch current at bus i towards bus j of the generic branch ij:

$$\bar{I}_{ij} = \bar{Y}_i \bar{V}_i + \bar{Y}_{ij}(\bar{V}_i - \bar{V}_j)$$



Therefore, we have:

$$\frac{\partial \bar{I}_{ij}}{\partial P_l} = \bar{Y}_i \frac{\partial \bar{V}_i}{\partial P_l} + \bar{Y}_{ij}\left(\frac{\partial \bar{V}_i}{\partial P_l} - \frac{\partial \bar{V}_j}{\partial P_l}\right)$$

$$\frac{\partial \bar{I}_{ij}}{\partial Q_l} = \bar{Y}_i \frac{\partial \bar{V}_i}{\partial Q_l} + \bar{Y}_{ij}\left(\frac{\partial \bar{V}_i}{\partial Q_l} - \frac{\partial \bar{V}_j}{\partial Q_l}\right)$$

Linearized OPF

From the complex sensitivity coefficients of the branch currents, we can easily compute the sensitivity of the **branch current modules** as:

$$K_{P,I}^{il} = \frac{\partial \left| \bar{I}_{ij} \right|}{\partial P_l} = \frac{1}{\left| \bar{I}_{ij} \right|} \Re \left(\underline{I}_{ij} \frac{\partial \bar{I}_{ij}}{\partial P_l} \right)$$

$$K_{Q,I}^{il} = \frac{\partial \left| \bar{I}_{ij} \right|}{\partial Q_l} = \frac{1}{\left| \bar{I}_{ij} \right|} \Re \left(\underline{I}_{ij} \frac{\partial \bar{I}_{ij}}{\partial Q_l} \right)$$

and **express the variations of the branch current modules** as:

$$\left[\mathbf{I}_{ij} \right] - \left[\mathbf{I}_{ij}^{*} \right] = \left[\mathbf{K}_{P,I} \right] \begin{bmatrix} P_2 - P_2^{*} \\ \vdots \\ P_s - P_s^{*} \end{bmatrix} + \left[\mathbf{K}_{Q,I} \right] \begin{bmatrix} Q_2 - Q_2^{*} \\ \vdots \\ Q_s - Q_s^{*} \end{bmatrix}$$

Where $\left[\mathbf{I}_{ij} \right]$ is the vector of the branch currents modules and $\left[\mathbf{I}_{ij}^{*} \right]$ the same vector computed in **correspondence of the state of the grid**.

Linearized OPF

Therefore, we can write the approximated linear OPF **(L-OPF)**as follows

$$\min_{P_{g_2},\dots,P_{g_s},Q_{g_2},\dots,Q_{g_s}} \sum_{i=1}^{3} C_i\left(P_{g_i}, Q_{g_i}\right)$$

$$s.t.$$

$$\begin{bmatrix} V_2 - V_2^* \\ \vdots \\ V_s - V_s^* \end{bmatrix} = \left[\mathbf{K}_{P,V}\right] \begin{bmatrix} P_2 - P_2^* \\ \vdots \\ P_s - P_s^* \end{bmatrix} + \left[\mathbf{K}_{Q,V}\right] \begin{bmatrix} Q_2 - Q_2^* \\ \vdots \\ Q_s - Q_s^* \end{bmatrix}$$

$$\left[\mathbf{I}_{ij}\right] - \left[\mathbf{I}_{ij}^*\right] = \left[\mathbf{K}_{P,I}\right] \begin{bmatrix} P_2 - P_2^* \\ \vdots \\ P_s - P_s^* \end{bmatrix} + \left[\mathbf{K}_{Q,I}\right] \begin{bmatrix} Q_2 - Q_2^* \\ \vdots \\ Q_s - Q_s^* \end{bmatrix}$$

$$P_i = P_{g_i} + P_{l_i}, i = 1, \dots, s$$

$$Q_i = Q_{g_i} + Q_{l_i}, i = 1, \dots, s$$

$$P_{g_i}^{min} \le P_{g_i} \le P_{g_i}^{max},\ i = 1, \dots, g$$

$$Q_{g_i}^{min} \le Q_{g_i} \le Q_{g_i}^{max},\ i = 1, \dots, g$$

$$\left|\overline{V}_1\right| = 1pu$$

$$V_{min} \le V_i \le V_{max}, i = 2, \dots, s$$

$$I_{ij} \le I_{i,j}^{max}, i \ne j = 1, \dots, s$$

The L-OPF contains also the constraints on the branch currents, it is linear and can be solved very efficiently. Voltage angles are missed.

Outline

Introduction

The DC approximation

Linearized OPF

Sequential linear programming OPF

Sequential linear programming OPF

The main problem of the an approximated OPF (including the linearized one) is that **the solution of the OPF does not strictly satisfies the load flow equations**. Therefore, the **constraints on nodal voltages and branch flows can still be violated if the OPF solution is implemented.**

To overcome this limitation, one can implement this heuristic algorithm:

Sequential linear programming (SLP) OPF

1. start from the knowledge of the system state $[\overline{\mathbf{V}}^*]^0$ (either from a state estimator or from a load flow solved with initial boundary conditions).
2. Do $h = 1 \ldots$
3. Compute matrices $\left[\mathbf{K}_{P,V}\right]^h, \left[\mathbf{K}_{Q,V}\right]^h, \left[\mathbf{K}_{P,I}\right]^h, \left[\mathbf{K}_{Q,I}\right]^h$ as fcn of $[\overline{\mathbf{V}}^*]^{h-1}$
4. Solve the problem L-OPF to obtain the nodal power injections $\left[\overline{\mathbf{S}}\right]^h$
5. Compute the load flow using the solution obtained from the L-OPF, $\left[\overline{\mathbf{S}}\right]^h$, to obtain a new system state $[\overline{\mathbf{V}}^*]^h$
6. Iterate until $\left\|[\overline{\mathbf{V}}^*]^h - [\overline{\mathbf{V}}^*]^{h-1}\right\| < \varepsilon_V$ and/or $\left\|[\overline{\mathbf{S}}^*]^h - [\overline{\mathbf{S}}^*]^{h-1}\right\| < \varepsilon_S$

Sequential linear programming OPF

In summary, the SLP-OPF works as follows: at line 4 it computes a L-OPF and, since the L-OPF is convex, we get a global minimum of the L-OPF. However, the output of the L-OPF cannot exactly satisfy the load flow equations since the L-OPF relies on sensitivity coefficients.

Therefore, we compute at line 5 a new load flow using the L-OPF (optimally computed) nodal power injections to obtain a new state of the grid. This updated state is used to re-compute the sensitivities at line 3 and solve the L-OPF with a linearization that is closer to satisfy the load flow equations exactly.

This process is iterated until the convergence of the residuals computed for the nodal voltages and/or the nodal injected powers (line 6).

It is important to note that this process is heuristic in the sense that it does not necessarily achieve the global optimum of the non-approximated original (and non-convex) OPF.

The user can decide to adopt the L-OPF or SLP-OPF as a function of the computation time associated to the two problems. In general, the L-OPF is more suitable for real-time control while the SLP-OPF for off-line computations.